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On the Curvature of Space†

By A. Friedman in Petersburg *

With one figure. Received on 29. June 1922

§1. 1. In their well-known works on general cosmological questions, Einstein[1] and de Sitter[2] arrive at two possible types of the universe; Einstein obtains the so-called cylindrical world, in which space[3] has constant, time-independent curvature, where the curvature radius is connected to the total mass of matter present in space; de Sitter obtains a spherical world in which not only space, but in a certain sense also the world can be addressed as a world of constant curvature.[4] In doing so both Einstein and de Sitter make certain presuppositions about the matter tensor, which correspond to the incoherence of matter and its relative rest, i.e. the velocity of matter will be supposed to be sufficiently small in comparison to the fundamental velocity[5] — the velocity of light.

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1 Einstein, Cosmological considerations relating to the general theory of relativity, *Sitzungsberichte Berl. Akad.* 1917.

2 de Sitter, On Einstein's theory of gravitation and its astronomical consequences. *Monthly Notices of the R. Astronom. Soc.* 1916-1917.

3 By "space" we understand here a space that is described by a manifold of three dimensions; the "world" corresponds to a manifold of four dimensions.

4 Klein, On the integral form of the conservation theorems and the theory of the spatially closed world. *Götting. Nachr.* 1918.

5 See this name by Eddington given in his book: *Espace, Temps et Gravitation*, 2 Partie, S. 10. Paris 1921.

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The goal of this Notice is, firstly the derivation of the cylindrical and spherical worlds (as special cases) from some general assumptions, and secondly the proof of the possibility of a world whose space curvature is constant with respect to three coordinates that serve as space coordinates, and dependent on the time, i.e. on the fourth — the time coordinate; this new type is, as concerning its other properties, an analogue of the Einstein cylindrical world.

2. The assumptions on which we base our considerations divide into two classes. To the first class belong assumptions which coincide with Einstein's and de Sitter's assumptions; they relate to the equations which the gravitational potentials obey, and to the state and the motion of matter. To the second class belong assumptions on the general, so to speak geometrical character of the world; from our hypothesis follows as a special case Einstein's cylindrical world and also de Sitter's spherical world.

The assumptions of the first class are the following:

1. The gravitational potentials obey the Einstein equation system with the cosmological term, which may also be set to zero:

$$R_{ik} - 1/2\, g_{ik} \overline{R} + \lambda g_{ik} = -\kappa T_{ik} \qquad (i, k = 1, 2, 3, 4), \tag{A}$$

here g_{ik} are the gravitational potentials, T_{ik} the matter tensor, κ — a constant, $\overline{R} = g^{ik} R_{ik}$; R_{ik} is determined by the equations

$$R_{ik} = \frac{\partial^2 lg\sqrt{g}}{\partial x_i\, \partial x_k} - \frac{\partial lg\sqrt{g}}{\partial x_\sigma} \left\{ {ik \atop \sigma} \right\} - \frac{\partial}{\partial x_\sigma} \left\{ {ik \atop \sigma} \right\} + \left\{ {i\alpha \atop \sigma} \right\} \left\{ {k\sigma \atop \alpha} \right\}, \tag{B}$$

where x_i $(i = 1, 2, 3, 4)$ are the world coordinates, and $\left\{ {ik \atop l} \right\}$ the Christoffel symbols of the second kind.[6]

2. Matter is incoherent and at relative rest; or, less strongly expressed, the relative velocities of matter are vanishingly small in comparison to the velocity of light. As a consequence of these assumptions the matter tensor is given by the equations:

$$\begin{aligned} T_{ik} &= 0 \quad \text{for } i \text{ and } k \text{ not} = 4, \\ T_{44} &= c^2 \rho g_{44}\,, \end{aligned} \tag{C}$$

where ρ is the density of matter and c the fundamental velocity; furthermore the world coordinates are divided into three space coordinates x_1, x_2, x_3 and the time coordinate x_4.

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[6] The sign of R_{ik} and of $\overline{R}$ is different in our case from the usual one.

3. The assumptions of the second class are the following:

I. After distribution of the three space coordinates x_1, x_2, x_3 we have a space of constant curvature, that however may depend on x_4 — the time coordinate. The interval[7] ds, determined by $ds^2 = g_{ik}dx_i dx_k$, can be brought into the following form through introduction of suitable space coordinates:

$$ds^2 = R^2(dx_1^2 + \sin^2 x_1\, dx_2^2 + \sin^2 x_1 \sin^2 x_2\, dx_3^2) \\ + 2g_{14}\, dx_1 dx_4 + 2g_{24}\, dx_2 dx_4 + 2g_{34}\, dx_3 dx_4 + g_{44}\, dx_4^2 .$$

Here R depends only on x_4; R is proportional to the curvature radius of space, which therefore may change with time.

2. In the expression for ds^2, g_{14}, g_{24}, g_{34} can be made to vanish by corresponding choice of the time coordinate, or, shortly said, time is orthogonal to space. It seems to me that no physical or philosophical reasons can be given for this second assumption; it serves exclusively to simplify the calculations. One must also remark that Einstein's and de Sitter's world are contained as special cases in our assumptions.

In consequence of assumptions 1 and 2, ds^2 can be brought into the form

$$ds^2 = R^2(dx_1^2 + \sin^2 x_1\, dx_2^2 + \sin^2 x_1 \sin^2 x_2\, dx_3^2) + M^2\, dx_4^2 , \qquad (D)$$

where R is a function of x_4 and M in the general case depends on all four world coordinates. The Einstein universe is obtained if one replaces in (D) R^2 by $-\frac{R^2}{c^2}$ and furthermore sets M equal to 1, whereby R means the constant (independent of x_4) curvature radius of space. De Sitter's universe is obtained if one replaces in (D) R^2 by $-\frac{R^2}{c^2}$ and M by $\cos x_1$:

$$d\tau^2 = -\frac{R^2}{c^2}(dx_1^2 + \sin^2 x_1\, dx_2^2 + \sin^2 x_1 \sin^2 x_2\, dx_3^2) + dx_4^2 , \qquad (D_1)$$

$$d\tau^2 = -\frac{R^2}{c^2}(dx_1^2 + \sin^2 x_1\, dx_2^2 + \sin^2 x_1 \sin^2 x_2\, dx_3^2) + \cos^2 x_1\, dx_4^2 . \qquad (D_2)$$ [8]

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[7] See e.g. Eddington, *Espace, Temps et Gravitation*, 2 Partie. Paris 1921.

[8] The ds, which is assumed to have the dimension of time, we denote by $d\tau$; then the constant κ has the dimension $\frac{\text{Length}}{\text{Mass}}$ and in CGS-units is equal to $1,87.10^{-27}$. See Laue, *Die Relativitätstheorie*, Bd. II, S. 185. Braunschweig 1921.

4. Now we must make an agreement over the limits within which the world coordinates are confined, i.e. over which points in the four-dimensional manifold we will address as being different; without engaging in a further justification, we will suppose that the space coordinates are confined within the following intervals: x_1 in the interval $(0, \pi)$; x_2 in the interval $(0, \pi)$ and x_3 in the interval $(0, 2\pi)$; with respect to the time coordinate we make no preliminary restrictive assumption, but will consider this question further below.

§2. 1. It follows from the assumptions (C) and (D), if one sets in equations (A) $i = 1, 2, 3$ and $k = 4$, that:

$$R'(x_4)\frac{\partial M}{\partial x_1} = R'(x_4)\frac{\partial M}{\partial x_2} = R'(x_4)\frac{\partial M}{\partial x_3} = 0\,;$$

from this there arise the two cases: (1.) $R'(x_4) = 0$, R is independent of x_4, we wish to designate this world as a stationary world; (2.) $R'(x_4)$ not $= 0$, M depends only on x_4; this shall be called the non-stationary world.

We consider first the stationary world and write down the equations (A) for $i, k = 1, 2, 3$ and further i not $= k$, thus we obtain the following system of formulae:

$$\begin{aligned}
\frac{\partial^2 M}{\partial x_1\,\partial x_2} - \operatorname{cotg} x_1\frac{\partial M}{\partial x_2} &= 0,\\
\frac{\partial^2 M}{\partial x_1\,\partial x_3} - \operatorname{cotg} x_1\frac{\partial M}{\partial x_3} &= 0,\\
\frac{\partial^2 M}{\partial x_2\,\partial x_3} - \operatorname{cotg} x_2\frac{\partial M}{\partial x_3} &= 0\,.
\end{aligned}$$

Integration of these equations yields the following expression for M:

$$M = A(x_3, x_4)\sin x_1 \sin x_2 + B(x_2, x_4)\sin x_1 + C(x_1, x_4), \tag{1}$$

where A, B, C are arbitrary functions of their arguments. If we solve the equations (A) for R_{ik} and eliminate the unknown density ρ[9] from the still unused equations, we obtain, if we substitute for M the expression (1), after somewhat lengthy, but completely elementary calculations the following two possibilities for M:

$$M = M_0 = \text{const}, \tag{2}$$

[9] The density ρ is in our case an unknown function of the world coordinates x_1, x_2, x_3, x_4.

$$M = (A_0 x_4 + B_0) \cos x_1, \tag{3}$$

where M_0, A_0, B_0 denote constants.

If M is equal to a constant, the stationary world is the cylindrical world. Here it is advantageous to operate with the gravitational potentials of formula (D_1); if we determine the density and the quantity λ, Einstein's well-known result will be obtained:

$$\lambda = \frac{c^2}{R^2}, \qquad \rho = \frac{2}{\kappa R^2}, \qquad \overline{M} = \frac{4\pi^2}{\kappa} R,$$

where $\overline{M}$ means the total mass of space.

In the second possible case, when M is given by (3), we arrive by means of a well-behaved transformation of x_4[10] at de Sitter's spherical world, in which $M = \cos x_1$; with the help of (D_2) we obtain de Sitter's relations:

$$\lambda = \frac{3c^2}{R^2}, \qquad \rho = 0, \qquad \overline{M} = 0\,.$$

Thus we have the following result: the stationary world is either the Einstein cylindrical world or the de Sitter spherical world.

2. We now want to consider the non-stationary world. M is now a function of x_4; by suitable choice of x_4 (without harming the generality of the consideration), one can obtain that $M = 1$; to relate this to our usual picture, we give ds^2 a form that is analogous to (D_1) and (D_2):

$$d\tau^2 = -\frac{R^2(x_4)}{c^2}\left(dx_1^2 + \sin^2 x_1\, dx_2^2 + \sin^2 x_1 \sin^2 x_2\, dx_3^2\right) + dx_4^2\,. \tag{D_3}$$

Our task is now the determination of R and ρ from the equations (A). It is clear that the equations (A) with differing indices give nothing; the equations (A) for $i = k = 1, 2, 3$ give one relation:

$$\frac{R'^2}{R^2} + \frac{2RR''}{R^2} + \frac{c^2}{R^2} - \lambda = 0, \tag{4}$$

the equation (A) with $i = k = 4$ gives the relation:

$$\frac{3R'^2}{R^2} + \frac{3c^2}{R^2} - \lambda = \kappa c^2 \rho, \tag{5}$$

[10] This transformation is given by the formula $d\overline{x}_4 = \sqrt{A_0 x_4 + B_0}\, dx_4$.

with

$$R' = \frac{dR}{dx_4} \qquad \text{and} \qquad R'' = \frac{d^2R}{dx_4^2}\,.$$

Since R' is not $= 0$, the integration of equation (4), when we also write t for x_4, gives the following equation:

$$\frac{1}{c^2}\left(\frac{dR}{dt}\right)^2 = \frac{A - R + \frac{\lambda}{3\,c^2}R^3}{R}\,, \tag{6}$$

where A is an arbitrary constant. From this equation we obtain R by inversion of an elliptic integral, i.e. by solving for R the equation

$$t = \frac{1}{c}\int_a^R \sqrt{\frac{x}{A - x + \frac{\lambda}{3\,c^2}x^3}}\;dx + B\,, \tag{7}$$

in which B and a are constants, and where also care must be taken of the usual conditions of the change of sign of the square root. From the equation (5), ρ can be determined to be

$$\rho = \frac{3\,A}{\kappa\,R^3}\,; \tag{8}$$

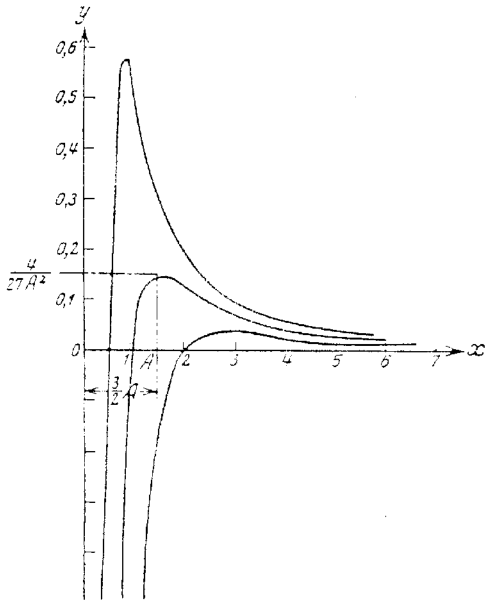
the constant A is expressed in terms of the total mass of space $\overline{M}$ according to:

$$A = \frac{\kappa\,\overline{M}}{6\,\pi^2}\,. \tag{9}$$

If $\overline{M}$ is positive, then also A will be positive.

3. We must base the consideration of the non-stationary world on equations (6) and (7); here the quantity λ is not determined; we will assume that it can have arbitrary values. We determine now those values of the variable x at which the square root of formula (7) can change its sign. Restricting our considerations to positive curvature radii, it suffices to consider for x the interval $(0, \infty)$ and within this interval those values of x that make the quantity under the square root equal to 0 or ∞. One value of x for which the square root in (7) is equal to zero, is $x = 0$; the remaining values of x, at which the square root in (7) changes its sign, are given by the positive roots of the equation $A - x + \frac{\lambda}{3\,c^2}x^3 = 0$. We denote $\frac{\lambda}{3\,c^2}$ by y and consider in the (x, y)-plane the family of curves of third degree:

$$yx^3 - x + A = 0\,. \tag{10}$$



A is here the parameter of the family, that varies in the interval $(0, \infty)$. The curves of the family (s. Fig.) cut the x-axis at the point $x = A$, $y = 0$ and have a maximum at the point

$$x = \frac{3A}{2}, \qquad y = \frac{4}{27A^2}.$$

From the figure it is apparent that for negative λ the equation $A - x + \frac{\lambda}{3c^2}x^3 = 0$ has one positive root x_0 in the interval $(0, A)$. Considering x_0 as a function of λ and A:

$$x_0 = \Theta(\lambda, A),$$

one finds that Θ is an increasing function of λ and an increasing function of A. If λ is located in the interval $\left(0, \frac{4}{9}\frac{c^2}{A^2}\right)$, the equation has two positive roots $x_0 = \Theta(\lambda, A)$ and $x_0' = \vartheta(\lambda, A)$, where x_0 falls in the interval $\left(A, \frac{3A}{2}\right)$ and x_0' in the interval $\left(\frac{3A}{2}, \infty\right)$; $\Theta(\lambda, A)$ is an increasing function both of λ and of A, $\vartheta(\lambda, A)$ a decreasing function of λ and A. Finally if λ is greater than $\frac{4}{9}\frac{c^2}{A^2}$, the equation has no positive roots.

We now go over to the consideration of formula (7) and precede this consideration by the following remark: let the curvature radius be equal to R_0 for $t = t_0$; the sign of the square root in (7) for $t = t_0$ is positive or negative according to whether for $t = t_0$ the curvature radius is increasing or decreasing; by replacing t by $-t$ if necessary, we can always make the square root positive, i.e. by choice of the time we can always achieve that the curvature radius for $t = t_0$ increases with increasing time.

4. We consider first the case $\lambda > \frac{4}{9}\frac{c^2}{A^2}$, i.e. the case when the equation $A - x + \frac{\lambda}{3c^2}x^3 = 0$ has no positive roots. The equation (7) can then be written in the form

$$t - t_0 = \frac{1}{c}\int_{R_0}^{R}\sqrt{\frac{x}{A - x + \frac{\lambda}{3c^2}x^3}}\,dx, \tag{11}$$

where according to our remark, the square root is always positive. From this it follows that R is an increasing function of t; the positive initial value R_0 is free of any restriction.

Since the curvature radius cannot be less than zero, so it must, decreasing from R_0 with decreasing time t, reach the value zero at the instant t'. The time of growth of R from 0 to R_0 we want to call the time since the creation of the world;[11] this time t' is given by:

$$t' = \frac{1}{c}\int_{0}^{R_0}\sqrt{\frac{x}{A - x + \frac{\lambda}{3c^2}x^3}}\,dx. \tag{12}$$

We designate the world considered as a monotonic world of the first kind.

The time since the creation of the (monotonic) world (of the first kind), considered as a function of R_0, A, λ, has the following properties: 1. it grows with growing R_0; 2. it decreases, when A increases, i.e. the mass in space increases; 3. it decreases, when λ increases. If $A > \frac{2}{3}R_0$, then for an arbitrary λ the time that has flowed since the creation of the world is finite; if $A \leq \frac{2}{3}R_0$, then there can always be found a value of $\lambda = \lambda_1 = \frac{4c^2}{9A^2}$ such that as λ approaches this value, the time since the creation of the world increases without limit.

5. Now λ shall lie in the interval $\left(0, \frac{4c^2}{9A^2}\right)$; then the initial value of the curvature radius can lie in the intervals: $(0, x_0)$, (x_0, x_0'), (x_0', ∞). If R_0 falls into the interval (x_0, x_0'), then the square root in formula (7) is imaginary; a space with this initial curvature is impossible.

[11] The time since the creation of the world is the time that has flowed from that instant when the space was one point ($R = 0$) until the present state ($R = R_0$); this time may also be infinite.

We will dedicate the next section to the case when R_0 lies in the interval $(0, x_0)$; here we still consider the third case: $R_0 > x_0'$ or $R_0 > \vartheta(\lambda, A)$. By considerations which are analogous to the preceding ones, it can be shown that R is an increasing function of time, where R can begin with the value $x_0' = \vartheta(\lambda, A)$. The time that has elapsed from the instant when $R = x_0'$ until the instant which corresponds to $R = R_0$, we again call the time since the creation of the world. Let this be t', then it is

$$t' = \frac{1}{c} \int_{x_0'}^{R_0} \sqrt{\frac{x}{A - x + \frac{\lambda}{3c^2} x^3}}\, dx. \tag{13}$$

This world we call a monotonic world of the second kind.

6. We consider now the case when λ falls between the limits $(-\infty, 0)$. In this case if $R_0 > x_0 = \Theta(\lambda, A)$, so the square root in (7) becomes imaginary, the space with this R_0 is impossible. If $R_0 < x_0$, then the considered case is identical with the one that we left aside in the previous section. We thus suppose that λ lies in the interval $\left(-\infty, \frac{4c^2}{9A^2}\right)$ and that $R_0 < x_0$. Through known considerations[12] one can now show that R is a periodic function of t, with period t_π, we call this the world period; t_π is given through the formula:

$$t_\pi = \frac{2}{c} \int_0^{x_0} \sqrt{\frac{x}{A - x + \frac{\lambda}{3c^2} x^3}}\, dx. \tag{14}$$

The curvature radius varies thereby between 0 and x_0. We want to call this world the periodic world. The period of the periodic world increases, when we increase λ, and tends towards infinity, when λ tends towards the value $\lambda_1 = \frac{4c^2}{9A^2}$.

For small λ the period is represented by the approximation formula

$$t_\pi = \frac{\pi A}{c}. \tag{15}$$

With respect to the periodic world, two viewpoints are possible: if we regard two events to be coincident if their space coordinates coincide and

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[12] See e.g. Weierstrass, On a class of real periodic functions. *Monatsber. d. Königl. Akad. d. Wissensch.* 1866, and Horn, On the theory of small finite oscillations. *ZS. f. Math. und Physik* **47**, 400, 1902. In our case, the considerations of these authors must be altered appropriately; the periodicity in our case can be determined by elementary considerations.

the difference of the time coordinates is an integer multiple of the period, then the curvature radius increases from 0 to x_0 and then decreases to the value 0; the time of the world's existence is finite; on the other hand, if the time varies between $-\infty$ and $+\infty$ (i.e. we consider two events to be coincident only when not only their space coordinates but also their world coordinates coincide), then we arrive at a true periodicity of the space curvature.

7. Our knowledge is completely insufficient to carry out numerical calculations and to decide, which world our universe is; it is possible that the causality problem and the problem of the centrifugal force will illuminate these questions. It is left to remark that the "cosmological" quantity λ remains undetermined in our formulae, since it is an extra constant in the problem; possibly electrodynamical considerations can lead to its evaluation. If we set $\lambda = 0$ and $M = 5.10^{21}$ solar masses, then the world period becomes of the order of 10 billion years. But these figures can surely only serve as an illustration for our calculations.

Petrograd, 29. May 1922.