Cosmic neutrinos and a brief synopsis of decoupling of fundamental particles from the thermal bath

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Preface

This 16 week internship program, from March 16 to July 5, forms a part of the Master 1 course at Ecole Polytechnique. The topic of the research, mostly bibliographic deals with the decoupling phenomena of fundamental particles from thermal bath at the early stage of the Universe and searches to explore the modifications, namely in the magnitude of the predicted temperature of the Cosmic Neutrino wave Background, in short CNB, in case a fourth generation neutrino would exist. Such existence can be justified in order to account for the observed anomaly of the net effective degrees of freedom for the known neutrinos.

The motive behind such choice of subject, was to deal with several sub-domains of Physics at the same time, namely, Statistical Mechanics, Particle Physics, Cosmology and Quantum Field Theory. On the other hand it aimed at sneaking into the state of art research in the sector of High Energy Physics along with a steady preparation for the ensuing academic year, 2015-2016.

This internship started with a detailed bibliographic work on the birth and evolution of the Universe, articles and lecture notes used has been cited in the section of Bibliography.

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Introduction

The origin of the Universe is attributed to a massive and robust explosion, the Big Bang, which is, in mathematical perspective, viewed as a space-time singularity. Just after the Big Bang, the Universe underwent a rapid expansion of very short duration, termed inflation. As time passes by, the Universe undergoes expansion, it cools down, the most fundamental constituents which so far shared the same thermal bath, or rather the thermal soup, start decoupling from the initial plasma. Initial quantum fluctuations were gigantically magnified in course of inflation which led to the formation of galactic structures.
The early Universe was hot and dense.Interactions between the fundamental particles in the thermal soup were frequent and energetic. Matter existed in the form of free leptons and atomic nuclei, or more generally bounded states of decoupled quarks with light bouncing between them. As the primordial plasma cooled, the light elements, namely, hydrogen, helium and lithium were formed. At some point, the energy had dropped enough for the first stable atoms to exist. At that moment, photons started to stream freely. Today, billions of years later, we observe this afterglow of the Big Bang as microwave radiation, termed Cosmic Microwave Background (CMB). This radiation is found to be almost completely uniform, the same temperature (about 2.73 K) in all directions. Crucially, the cosmic microwave background contains small variations in temperature at a level of 1 part in 10 000. Parts of the sky are slightly hotter, parts slightly colder. These fluctuations reflect tiny variations in the primordial density of matter. Over time, and under the influence of gravity, these fluctuations grew. Dense regions were getting denser. Eventually, galaxies, stars and planets were formed.
This picture of the universe,from fractions of a second after the Big Bang until today, is a scientific fact. However, the story isn't without surprises. The majority of the universe today consists of forms of matter and energy that are unlike anything we have ever seen in terrestrial experiments. Dark matter is required to explain the stability of galaxies and the rate of formation of large-scale structures. Dark energy is required to rationalize the striking fact that the expansion of the universe started to accelerate recently (meaning a few billion years ago). What dark matter and dark energy are is still a mystery. Finally, there is growing evidence that the primordial density perturbations originated from microscopic quantum fluctuations, stretched to cosmic sizes during a period of inflationary expansion. The physical origin of inflation is still a topic of active research.

1 Few Preliminary Notions

1.1 Space time metric of the Universe

At large scale, the Universe appears to be homogeneous and isotropic. This indicates the possibility to represent the Universe by a time-ordered sequence of three-dimensional spatial slices Σ_t, each of which is homogeneous and isotropic. A homogeneous and isotropic $3-$ *space* must be dotted with a constant $3-$ *curvature*. Such maximally symmetric $3-$ *space* can be classified to:

flat space with zero curvature, bearing the line element,

$$dl^2 = d\mathbf{x}^2$$

and is invariant under translation,

positively curved space with a constant positive curvature, a, can be represented as a $3-$ *sphere* embedded in a $4-d$ Euclidean space and with the line element,

$$dl^2 = d\mathbf{x}^2 + du^2, \quad with\ \mathbf{x}^2 + u^2 = a^2$$

negatively curved space with a constant negative curvature, a, can be represented as a *hyperboloid* embedded in a $4-d$ Lorentzian space having the line element,

$$dl^2 = d\mathbf{x}^2 - du^2, \quad where\ \mathbf{x}^2 - u^2 = -a^2$$

.

Rescaling the coordinates,

$$\mathbf{x} \to a\mathbf{x}\ and\ u \to au,$$

we obtain the line element unifying the above three scenarios,

$$dl^2 = a^2\left[d\mathbf{x}^2 + k\frac{(\mathbf{x}.d\mathbf{x})^2}{1-kx^2}\right] = a^2\gamma_{ij}dx^idx^j,$$

with

$$\gamma_{ij} = \delta_{ij} + k\frac{x_ix_j}{1-k(x^kx_k)}$$

and where $k = 0$ for Euclidean flat space, 1 for the spherical space and -1 for the hyperbolic space.
Such rescaling renders the coordinates $\mathbf{x}$ and u dimensionless, the parameter a has the dimension of length.

In spherical coordinate system, (r, θ, ϕ), the above expression reduces to,

$$dl^2 = a^2\left[\frac{dr^2}{1-kr^2} + r^2d\Omega^2\right],$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

Robertson Walker Metric

In order to obtain Robertson Walker metric for an expanding Universe, we replace the spatial part of the Minkowski metric with the line element obtained above, letting a to be a function of time, thus the metric bears the form,

$$ds^2 = dt^2 - a^2(t)\gamma_{ij}dx^i dx^j$$

which in polar coordinates, reads

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

We notice that the above metric has a rescaling symmetry,

$$a \leftarrow \lambda a,\ r \leftarrow r/\lambda,\ k \leftarrow \lambda^2 k$$

This signifies that the geometry of the spacetime stays the same if we simultaneously rescale a, r, k as above.

The coordinates $x^i = (x^1, x^2, x^3)$ are called *comoving coordinates*. The *physical coordinates* are given by, $x^i_{phy} = a(t)x^i$. The physical velocity is given by,

$$v^i_{phy} = \frac{dx^i_{phy}}{dt} = a(t)\frac{dx^i}{dt} + \frac{da}{dt}(t)x^i = v^i_{pec} + H(t)x^i_{phy}$$

It has two contributions, the so called *peculiar velocity*, $v^i_{pec} = a(t)\dot{x}$ and the *Hubble flow*, Hx^i_{phy}, where the *Hubble parameter* is defined as,

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

1.2 Cosmic Inventory

The universe contains a mixture of different matter components, it seems useful to classify them according to their contribution to the pressure.
Let us get started from the continuity equation,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{1}$$

Matter

The term "matter" is referred to all forms of matter for which the pressure is much smaller than the energy density, $|P| << \rho$. This is precisely the case for a gas of non-relativistic particles (where the energy density is dominated by the mass). Setting $P = 0$ in eqn.(1), we obtain,

$$\rho \propto a^{-3}$$

Such dilution of the energy density simply reflects the expansion of the volume,

$$V \propto a^3$$

Dark Matter

Most of the matter in the Universe exists in the form of Dark Matter, which can only be perceived by their gravitational effects, they are thought to be comprised of yet unknown heavy particle species which do not interact neither electromagnetically, nor strongly, nor weakly.

Baryons

Cosmologists refer the term baryons to the ordinary matter (eg. nuclei and electrons). In a scrupulous way, this is technically incorrect since the leptons do not fall into this category, but the nuclei are so much heavier than leptons that most of the mass is in the baryons and it is a reasonable approach to refer them altogether as baryons.

Radiation

The term "radiation" denotes anything for which the pressure is about a third of the energy density, $P = \frac{1}{3}\rho$. This is the case for a gas of relativistic particles, for which the energy density is dominated by the kinetic energy (i.e. the momentum is much bigger than the mass). In this case, eqn.(1) gives

$$\rho \propto a^{-4}$$

The dilution now includes the redshifting of the energy,

$$E \propto a^{-1}.$$

Photons

The early universe was dominated by photons. Being massless, they are always relativistic. Today, we detect those photons in the form of the cosmic microwave background.

Neutrinos

For most of the history of the universe, neutrinos behaved like radiation. Only recently have their small masses become relevant and they started to behave like matter.

Gravitons

The early universe is thought to have produced a background of gravitons (i.e. gravitational waves). Experimental efforts are underway to detect them.

Dark Energy

Our recent observations have confirmed that matter and radiation aren't enough to describe the evolution of the universe. Instead, the universe today seems to be dominated by a mysterious *negative* pressure component, $P = -\rho$, unlikely to anything so far encountered. More precisely from eqn.(1), we notice that the energy density of the dark matter is constant,

$$\rho \propto a^0$$

Since the energy density doesn't dilute, energy has to be created as the universe expands.

1.3 Critical densities and dimensionless density parameters

Let's start with the **Friedmann Equations**,

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{3}$$

where ρ and P should be understood as the sum of all contributions to the energy density and pressure in the universe.
We write ρ_r for the contribution from radiation (with ρ_γ for photons and ρ_ν for the neutrinos), ρ_m for the contribution by matter (with ρ_c for cold dark matter andρ_b for baryons) and ρ_Λ for vacuum energy contribution.
Using the subscript '$0'$' to denote the quantities evaluated today (at $t = t_0$), the present critical density for a flat Universe ($k = 0$) corresponds to,

$$\rho_{crit,0} = \frac{3H_0^2}{8\pi G}$$

Using the critical density let us define dimensionless density parameters,

$$\Omega_{I,0} = \frac{\rho_{I,0}}{\rho_{crit,0}}$$

Eqn.(2) can then be written as,

$$H^2(a) = H_0^2\left[\Omega_{r,0}\left(\frac{a_0}{a}\right)^4 + \Omega_{m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{k,0}\left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda,0}\right] \tag{4}$$

where a "curvature" density parameter is defined as, $\Omega_{k,0} = -k/(a_0 H_0)^2$.

Observations show that the universe is filled with radiation ('r'), matter ('m') and dark energy ('Λ'):

$$|\Omega_k| \leq 0.01, \quad \Omega_r = 9.4\ 10^{-5}, \quad \Omega_m = 0.32, \quad \Omega_\Lambda = 0.68$$

with

$$\Omega_b = 0.05, \quad \Omega_c = 0.27.$$

1.4 Single-Component Universe

Most cosmological fluids can be parameterized in terms of a constant equation of state: $w = P/\rho$, this includes cold dark matter ($w = 0$), radiation ($w = 1/3$) and vacuum energy ($w = -1$) and thus solution to the eqn.(1) reads as

$$\rho \propto a^{-3(1+w)}$$

The different scalings of radiation (a^{-4}), matter ($a^{-3}3$) and vacuum energy (a^0) imply that for most of its history the universe was dominated by a single component (first radiation, then matter, then vacuum energy). A parametrization of this component by its equation of state w_I captures all cases of interest. For a flat single-component Universe, with $\Omega_k \sim 0$ and $a_0 = 1$, eqn.(3) reads,

$$\frac{\dot{a}}{a} = H_0 \sqrt{\Omega_I} a^{-\frac{3}{2}(1+w_I)}$$

This upon integration gives the time dependence of the scale factor,

$$\begin{aligned} a(t) &\propto t^{2/3} \qquad for\ matter\ dominated\ Universe \\ &\propto t^{1/2} \qquad for\ radiation\ dominated\ Universe \\ &\propto e^{Ht} \qquad for\ darkenergy\ dominated\ Universe \end{aligned}$$

2 Thermal History and Decoupling Phenomena

Our main concern in this section is the evolution of the Universe, following the inflation, starting from the hot and dense state. At early times, the thermodynamical properties of the universe were determined by local equilibrium, the energy density ρ and the temperature T of the particles in the thermal soup were high, and the constituting particles behaved like a perfect plasma gas. Expansion of the Universe set in, decreasing gradually ρ and T, the conservation principle of entropy imposes the following constraint on the evolution of temperature,

$$T(t)a(t) = T_0 a_0$$

The evolution of the thermal bath for the first couple of seconds following the birth of the Universe was very slow, with reasonable accuracy all the contemporary processes in the soup are considered reversible, with adiabatic evolution. As the temperature decreases, so does the kinetic energy of the particles ($E_{kin} \sim \frac{3}{2}k_B T$), the particle no longer remained relativistic, their density decreased, having experienced lesser and lesser interactions with the primordial plasma, the particle decoupled from the thermal bath containing photons, which no longer possessed sufficient energy to create the concerned particle-antiparticle pairs.
In course of time, non-equilibrium processes sets in, the non-equilibrium dynamics allows massive particles to acquire cosmological abundances and therefore explains why there is something rather than nothing, it is equally crucial for understanding the origin of the cosmic microwave background and the formation of the light chemical elements.

The Hot Big Bang

The key notion for the understanding of the thermal history of the universe is the comparison between the rate of interactions Γ and the rate of expansion H. When $\Gamma >> H$, the time scale of particle interactions is much smaller than the characteristic expansion time scale,

$$t_c \equiv \frac{1}{\Gamma} << t_H \equiv \frac{1}{H} \tag{5}$$

Local thermal equilibrium is then reached before the effect of the expansion becomes relevant. As the universe cools, the rate of interactions may decrease faster than the expansion rate. At $t_c \sim t_H$, the particles decouple from the thermal bath. Different particle species may have different interaction rates and so may decouple at different times.

2.1 Local Thermal Equilibrium

Let us start with verifying that, the condition (5) is satisfied for Standard Model processes at temperatures above a few hundred GeV. The rate of particle interactions reads as,

$$\Gamma \equiv n\sigma v$$

where n is the number density of particles, σ is their interaction cross section, and v is the average velocity of the particles.

For $\Gamma \gtrsim 100 Gev$ all known particles are ultra-relativistic, hence $v \sim c \sim 1$, the particle density, $n \sim T^3$. Interactions are mediated by gauge bosons, which are massless above the scale of electroweak symmetry breaking. The cross sections for the strong and electroweak interactions then have a similar dependence, which at the lowest order of perturbation reads,

$$\sigma \sim \frac{\alpha^2}{T^2}$$

where $\alpha \equiv g_A^2/4\pi$ is the generalized structure constant associated with the gauge boson A. We finally obtain,

$$\Gamma = n\sigma v \equiv T^3 \frac{\alpha^2}{T^2} = \alpha^2 T$$

We compare the above magnitude with Hubble expansion rate, given by, $H \sim \sqrt{\rho}/M_{Pl}$. With $\rho \sim T^4$ in the ultra-relativistic regime we have, $H \sim \frac{T^2}{M_{Pl}^2}$ and the ratio of the above two reads,

$$\frac{\Gamma}{H} \sim \frac{\alpha^2 M_{Pl}}{T} \sim \frac{10^{16} Gev}{T}$$

where we have used $\alpha \sim 0.01$ in the numerical estimate. Below $T \sim 10^{16} Gev$ but above $100 Gev$ the condition(5) is therefore satisfied.
As the particles rests in equilibrium, they reach a state of maximum entropy upon exchanging energy and momentum efficiently. It is a standard result of statistical mechanics that the average occupation of a state of energy E by the concerned particles, per unit volume in phase space-the distribution function-then assumes the form (neglecting the chemical potential),

$$f(E) = \frac{1}{e^{E/T} \pm 1} \tag{6}$$

where the + sign is for fermions and the − sign for bosons. When the temperature drops below the mass of the particles, $T << m$, they become non-relativistic and their distribution function receives an exponential suppression, $f \sim e^{-m/T}$. This means that relativistic particles ('*radiation*') dominate the density and pressure of the primordial plasma. The total energy density is therefore well approximated by summing over all relativistic particles, $\rho_r \propto \sum_i \int d^3p f_i(p) E_i(p)$, more precisely,

$$\rho_r = \frac{\pi^2}{30} g_*(T) T^4$$

where $g_*(T)$ is the number of relativistic degrees of freedom. It corresponds to the internal degrees of relativistic Standard Model particles in the thermal bath. The value of g_* decreases whenever the temperature of the universe drops below the mass of a particle species and it becomes non-relativistic. Today, only photons and (maybe) neutrinos are still relativistic and $g_* = 3.38$.

Freeze-out and Decoupling

If equilibrium had persisted until today, the universe would be mostly photons. Any massive particle species would be exponentially suppressed. To understand the world around us, it is therefore crucial to understand the deviations from equilibrium that led to the freeze-out of massive particles. Below the scale of electroweak symmetry breaking, $T \lesssim 100GeV$, the gauge bosons of the weak interactions, $W^{\pm}$ and Z, receive masses $M_W \sim M_Z$. The cross section associated with processes mediated by the weak force becomes, at the lowest order of the perturbation,

$$\sigma \sim G_F^2 T^2$$

where Fermi's constant, $G_F \sim \alpha/M_W^2 \sim 1.17\ 10^{-5} GeV^{-2}$. The strength of the weak interactions now decreases as the temperature of the universe drops. We notice that,

$$\frac{\Gamma}{H} \sim \frac{\alpha^2 M_{Pl} T^3}{M_W^4} \sim \left(\frac{T}{1MeV}\right)^3 \tag{7}$$

which drops below unity at $T_{dec} \sim 1MeV$. Particles that interact with the primordial plasma only through the *weak interaction* therefore decouple around $1MeV$. This decoupling of weak scale interactions has important consequences for the thermal history of the universe.

2.2 A brief overview of the thermal history of the Universe

Baryogeneis

Relativistic quantum field theory requires the existence of anti-particles, this poses a slight puzzle. Particles and antiparticles annihilate through processes such as $e^+ + e^- \to \gamma + \gamma$. If initially the universe was filled with equal amounts of matter and anti-matter then we expect these annihilations to lead to a universe dominated by radiation. However, we do observe an overabundance of matter (mostly baryons) over anti-matter in the universe today. Models of baryogenesis intend to derive the observed baryon-to-photon ratio,

$$\eta \equiv \frac{n_b}{n_\gamma} \sim 10^{-9}$$

from some dynamical mechanism, i.e. without assuming a primordial matter-antimatter asymmetry as an initial condition. Although many ideas for baryogenesis exist, none is singled out by experimental tests.

Electrweak Phase Transitions

At $100GeV$ particles receive their masses through the Higgs mechanism, this leads to a drastic change in the strength of the weak interaction.

QCD Phase Transition

While quarks are *asymptotically free* (i.e.weakly interacting) at high energies, below $150MeV$, the strong interactions between the quarks and the gluons become important. Quarks and gluons then form bound three-quark systems, called *baryons*, and quark-antiquark pairs, called *mesons*. These baryons and mesons are the relevant degrees of freedom below the scale of the QCD phase transition.

Dark-Matter freeze out
Since dark matter is very weakly interacting with ordinary matter, it is expected to decouple relatively early on. It can be shown that an appropriate choice of natural values for the mass of the dark matter particles and their interaction cross section with ordinary matter reproduces the observed relic dark matter density surprisingly well, called the WIMP miracle.

Neutrino decoupling
Neutrinos interact with the rest of the primordial plasma only through the weak interaction. Hence eqn.(7) applies to the case of neutrinos and they actually decoupled at $3.54MeV$.

Electron-Positron annihilation
Electrons and positrons annihilate shortly after neutrino decoupling. The energies of the electrons and positrons gets transferred to the photons, but not the neutrinos. This explains why the photon temperature today is greater than the neutrino temperature.

Big Bang Nucleosynthesis
Around 3 minutes after the Big Bang, the light elements were formed.

Recombination
Neutral hydrogen forms through the reaction $e^- + p^+ \rightarrow H + \gamma$ when the temperature has become low enough that the reverse reaction is energetically disfavored.

Photon decoupling
Before recombination the strongest coupling between the photons and the rest of the plasma is through Thomson scattering, $e^- + \gamma \rightarrow e^- + \gamma$. The sharp drop in the free electron density after recombination means that this process becomes inefficient and the photons decouple. They have since streamed freely through the universe and are today observed as the *cosmic microwave background* (CMB).

2.3 Equilibrium Thermodynamics

We begin our discussion recalling some basic facts of equilibrium thermodynamics, suitably generalized to apply to an expanding universe.
Let us consider a gas of weakly interacting particles, conveniently we describe the system in the *phase space* where the gas is described by the positions and momenta of all particles. As follows from the *particle in a box* notion in *Quantum Mechanics*, the density of states for a system of particles with number internal degrees of freedom, g reads, $g/(2\pi)^3$. A knowledge of how the particles are distributed amongst the momentum eigenstates would lead us to the desired number density. This information is contained in the (*phase space*) distribution function $f(\mathbf{x}, \mathbf{p}, t)$. Because of homogeneity, the distribution function should, in fact, be independent of the position $\mathbf{x}$. Moreover, isotropy requires that the

momentum dependence is only in terms of the magnitude of the momentum $|\mathbf{p}| \equiv p$. We will typically leave the time dependence implicit, it will manifest itself in terms of the temperature dependence of the distribution functions. The particle density in phase space is then the density of states times the distribution function,

$$\frac{g}{(2\pi)^3} f(p)$$

The *number density* of particles in the real space is then obtained upon integrating the above equation over momentum,

$$n = \frac{g}{(2\pi)^3} \int d^3p f(p)$$

To obtain the energy density of the gas of particles, we have to weight each momentum eigenstate by its energy. To a good approximation, the particles in the early universe were *weakly interacting*. This allows us to ignore the interaction energies between the particles and write the energy of a particle of mass m and momentum p simply as,

$$E(p) = \sqrt{p^2 + m^2}$$

The *energy density* then reads,

$$\rho = \frac{g}{(2\pi)^3} \int d^3p f(p) E(p)$$

and the *pressure*,

$$P = \frac{g}{(2\pi)^3} \int d^3p f(p) \frac{p^2}{3E(p)}$$

Equilibria and their varieties

A system of particles is said to be in *kinetic equilibrium* if the particles exchange energy and momentum efficiently. This leads to a state of maximum entropy in which the distribution functions are given by the *Fermi-Dirac* and *Bose-Einstein* distributions,

$$f(p) = \frac{1}{e^{(E(p)-\mu)/T} \pm 1} \tag{8}$$

where, + sign applies for the fermions and − for the bosons.At low temperatures, $T < E - \mu$, both distribution functions reduce to the *Maxwell-Boltzmann* distribution

$$f(p) \approx e^{-(E(p)-\mu)/T} \tag{9}$$

The equilibrium distribution functions have two parameters: the *temperature* T and the *chemical potential* μ. The chemical potential may be temperature-dependent. As the universe expands, T and $\mu(T)$ change in such a way that the continuity equations for the energy density ρ and the particle number density n are satisfied. Each particle species i (with possibly distinct m_i, μ_i, T_i) has its own distribution function f_i and hence its own n_i, ρ_i, and P_i.

Concerning the chemical potential, a knowledge of the chemical potential of reacting particles can be used to indicate which way a reaction proceeds. The second law of thermodynamics indicates that particles flow towards the side of the reaction with the lower total chemical potential. *Chemical equilibrium* is reached when the sum of the chemical potentials of the reacting particles is equal to the sum of the chemical potentials of the products. The rates of the forward and reverse reactions are then equal.

If a species i is in *chemical equilibrium*, then its chemical potential μ_i is related to the chemical potentials μ_j of the other species it interacts with. For example, if a species 1 interacts with species $2,3$ and 4 via the reaction, $1+2 \leftrightarrow 3+4$, then chemical equilibrium implies,

$$\mu_1 + \mu_2 = \mu_3 + \mu_4$$

Since the number of photons is not conserved (e.g. double Compton scattering $e^- + \gamma \rightarrow e^- + \gamma + \gamma$ happens in equilibrium at high temperatures),

$$\mu_\gamma = 0$$

This implies that if the chemical potential of a particle X is μ_X, then the chemical potential of the corresponding anti-particle $\bar{X}$ is

$$\mu_{\bar{X}} = -\mu_X,$$

(to convince ourselves, we may consider the particle-antiparticle annihilation, $X + \bar{X} \leftrightarrow \gamma + \gamma$)

Thermal equilibrium is achieved for species which are both in kinetic and chemical equilibrium. These species then share a common temperature $T_i = T$.

Evaluation of the thermodynamical parameters

At the early age of the Universe, the chemical potential was small enough to be neglected and still obtain the related results with reasonable accuracy. The number density and the energy density then reads,

$$n = \frac{g}{2\pi^2} \int_0^\infty dp \frac{p^2}{exp[\sqrt{p^2+m^2}/T] \pm 1} \tag{10}$$

$$\rho = \frac{g}{2\pi^2} \int_0^\infty dp \frac{p^2\sqrt{p^2+m^2}}{exp[\sqrt{p^2+m^2}/T] \pm 1} \tag{11}$$

Defining $x \equiv m/T$ and $\chi \equiv p/T$, we obtain,

$$n = \frac{g}{2\pi^2} T^3 I_\pm(x), \qquad I_\pm(x) = \int_0^\infty d\chi \frac{\chi^2}{exp[\sqrt{\chi^2+x^2}] \pm 1} \tag{12}$$

$$\rho = \frac{g}{2\pi^2} T^4 J_\pm(x), \qquad J_\pm(x) = \int_0^\infty d\chi \frac{\chi^2\sqrt{\chi^2+x^2}}{exp[\sqrt{\chi^2+x^2}] \pm 1} \tag{13}$$

In general, the functions $I_\pm(x)$ and $J_\pm(x)$ have to be evaluated numerically. However, in the (ultra)relativistic and non-relativistic limits, we can get analytical results.

In the *relativistic limit*, $x \rightarrow 0$ ($m << T$) and eqn.(12) reduces to,

$$I_\pm(0) = \int_0^\infty d\chi \frac{\chi^2}{e^\chi \pm 1},$$

for bosons, using the standard integral,

$$\int_0^\infty d\chi \frac{\chi^n}{e^\chi - 1} = \zeta(n+1)\Gamma(n+1)$$

we obtain,

$$I_-(0) = 2\zeta(3)$$

Noting,

$$\frac{1}{e^\chi + 1} = \frac{1}{e^\chi - 1} - \frac{2}{e^{2\chi} - 1}$$

we obtain,

$$I_+(0) = I_-(0) - 2\left(\frac{1}{2}\right)^3 I_-(0) = \frac{3}{4} I_-(0)$$

Hence, we finally arrive at,

$$n = \frac{\zeta(3)}{\pi^2} g T^3 \qquad for\ bosons \tag{14}$$

$$= \frac{3\zeta(3)}{4\pi^2} g T^3 \qquad for\ fermions \tag{15}$$

A similar computation leads to,

$$\rho = \frac{\pi^2}{30} g T^4 \qquad for\ bosons, \tag{16}$$

$$= \frac{7}{8}\frac{\pi^2}{30} g T^4 \qquad for\ fermions \tag{17}$$

Finally, using the expression for the pressure, we verify the expected pressure-density relation for a relativistic gas,

$$P = \frac{1}{3}\rho \tag{18}$$

Noting the present temperature of cosmic microwave background radiation, $T_0 = 2.73K$ we get the relic abundance and the energy density of the last scattered photons,

$$n_{\gamma,0} = \frac{2\zeta(3)}{\pi^2} T_0^3 \approx 410\ photons\ cm^{-3}$$

$$\rho_{\gamma,0} = \frac{\pi^2}{15} g T_0^4 \approx 4.6\ 10^{-34} g\ cm^{-3}$$

In the classical limit, $x >> 1$, $m >> T$ bosons and fermions follow identical distribution function,

$$I_\pm(x) \approx \int_0^\infty d\chi \frac{\chi^2}{e^{\sqrt{\chi^2 + x^2}}}$$

Since most of the contribution to the integral comes from $x >> \chi$, therefore,

$$I_\pm(x) \approx \int_0^\infty d\chi \frac{\chi^2}{e^{x + \chi^2/(2x)}} = e^{-x} \int_0^\infty d\chi \chi^2 e^{-\chi^2/(2x)} = (2x)^{3/2} e^{-x} \int_0^\infty d\chi \chi^2 e^{-\chi^2}$$

Using the standard integral, $\int_0^\infty d\chi \chi^n e^{-\chi^2} = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)$, and noting that, $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, we have,

$$I_\pm(x) = \sqrt{\frac{\pi}{2}} x^{3/2} e^{-x}$$

and this leads to,

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-mT} \qquad (19)$$

At lowest order in the non-relativistic limit, we have $E(p) \approx m$ and the energy density is simply equal to the mass density,

$$\rho \approx mn \qquad (20)$$

And, using the general expression for the pressure, we obtain,

$$P = nT << mn = \rho \qquad (21)$$

as, $m << T$. That is, a non-relativistic gas of particles acts like pressureless dust.

A comparison between the results in the relativistic limit ($T >> m$) and in the non-relativistic limit ($T << m$) shows that the number density, energy density, and pressure of a particle species fall exponentially (are "Boltzmann suppressed") as the temperature drops below the mass of the particle. We interpret this as the annihilation of particles and anti-particles. At higher energies these annihilations also occur, but they are balanced by particle-antiparticle pair production. At low temperatures, the thermal energies of particles are insufficient for pair production.

2.4 Relativistic species and the number of effective internal degrees of freedom

Next, we consider T the temperature of the photon gas. The total radiation density is the sum over the energy densities of all relativistic species,

$$\rho_{tot,r} = \sum_i \rho_i = \frac{\pi^2}{30} g_*(T) T^4 \qquad (22)$$

where $g_*(T)$ is the *effective number of relativistic degrees of freedom* at the temperature T. The sum over particle species may receive two types of contributions:

Relativistic species in thermal equilibrium with the photons, $T_i = T >> m_i$,

$$g_*^{th}(T) = \sum_{i=b} g_i + \frac{7}{8} \sum_{i=f} g_i$$

(When the temperature drops below the mass m_i of a particle species, it becomes nonrelativistic and is removed from the sum. Away from mass thresholds, $g_*(T)$ is independent of temperature).

Relativistic species that are not in thermal equilibrium with the photons, $T_i \neq T >> m_i$,

$$g_*^{dec}(T) = \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{i=f} g_i \left(\frac{T_i}{T}\right)^4$$

where the decoupled species may have different temperatures T_i. This is relevant for neutrinos after e^+e^- annihilation.

We next study the evolution of $g_*(T)$ assuming the Universe to be composed of only Standard Model particles. At $T \gtrsim 100\ GeV$ all the particles were relativistic. Adding up their internal degrees of freedom, we have,

$$g_b = 28 \quad photons(2),\ Vector\ bosons\ W^{\pm}\ and\ Z^0(3.3),\ gluons(2.8),\ Higgs\ boson(1)$$

and

$$g_f = 90 \qquad quarks(2.2.3.6),\ charged\ leptons(2.2.3),\ neutrinos(2.3)$$

and so,

$$g_* = g_b + \frac{7}{8} g_f = 106.75 \tag{23}$$

As the temperature drops, various particle species become non-relativistic and annihilate. To estimate g_* at a temperature T we simply add up the contributions from all relativistic degrees of freedom (with $m << T$) and discard the rest.

Kinetic decoupling of different Standard Model particles and evolution of g_*

Being the heaviest particles of the Standard Model, the top quarks annihilates first, roughly at $T \sim 30GeV$ and the effective number of degrees of freedom, g_* reduced to, $106.75 - \frac{7}{8}.12 = 96.25$

The Higgs boson and the gauge bosons $W^{\pm}, Z^0$ were the next to annihilate,roughly at the same time, at $T \sim 10GeV$. g_* reduced to $96.25 - (1 + 3.3) = 86.25$.

Then, the bottom quarks annihilate ($g_* = 86 : 25 - \frac{7}{8}.12 = 75.75$), followed by the charm quarks and the tau leptons ($g_* = 75.75 - \frac{7}{8}.(12 + 4) = 61.75$).

As per the hierarchy of mass, the strange quarks were the next to annihilate, but before that, the underwent the QCD phase transition. At $T \sim 150MeV$, the quarks combine into baryons (protons, neutrons, ...) and mesons (pions, ...). There are many different species of baryons and mesons, but all except the pions ($\pi^{\pm},\ \pi^0$) were non-relativistic below the temperature of the QCD phase transition. Thus, the only particle species left in large numbers were the pions, electrons, muons, neutrinos, and the photons. The three pions ($spin - 0$) contribute to g_*, a value of $3.1 = 3$. We are therefore left with $g_* = 2 + 3 + \frac{7}{8}.(4 + 4 + 6) = 17.25$.

Next the pions along with the muons decoupled from the thermal bath, reducing g_* to 10.75, followed by the decoupling of the electrons leaving behind the photons and the neutrinos as the only relativistic species.

2.5 Conservation of Entropy

In cosmology, entropy carries more information than the energy. According to the second law of thermodynamics, the total entropy of the universe only increases or stays constant. Starting from the expression, $TdS = PdV + dU$ and noting that $U = \rho V$, we notice that the entropy is conserved in the evolution of the Universe as long as it is in equilibrium (for that we use the standard result, *for a system in equilibrium, with* $\mu = 0$, $\frac{\partial P}{\partial T} = \frac{P+\rho}{T}$, and eqn.(1)).
The differential change in entropy then assumes the form,

$$\frac{dS}{dt} = \frac{d}{dt}\left[\frac{\rho + P}{T}V\right] = 0$$

.

Since the number of photons outnumbers by far the number of baryons in the universe, the entropy of the universe is dominated by the entropy of the photon bath (at least as long as the universe is sufficiently uniform). Any entropy production from non-equilibrium processes is therefore totally insignificant relative to the total entropy. To a good approximation we can therefore treat the expansion of the universe as adiabatic, so that the total entropy stays constant even beyond equilibrium.

For convenience, we define the entropy density to be, $s \equiv \frac{S}{V} = \frac{P+\rho}{T}$.
Taking into account, the Boltzmann suppression of non-relativistic species,which have already undergone kinetic decoupling from the thermal bath, we obtain the total entropy density for a collection of different particle species, using eqns.(16) and (18)

$$s = \sum_i \frac{\rho i + Pi}{Ti} \equiv \frac{2\pi^2}{45} g_{*S}(T)T^3 \tag{24}$$

we define g_{*S} to be the *effective number of degrees of freedom in entropy*,

$$g_{*S}(T) = g_{*S}^{ther}(T) + g_{*S}^{dec}(T)$$

For relativistic species in thermal equilibrium with the photon gas, $g_{*S}^{ther}(T) = g_*^{ther}(T)$. But since, $s_i \propto Ti^3$,

$$g_{*S}^{dec}(T) \equiv \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{i=f} g_i \left(\frac{T_i}{T}\right)^3$$

Thus, g_{*S} is equal to g_* only when all the relativistic species are in equilibrium at the same temperature. In the real universe, this condition was held till $t \sim 1$ sec .

The conservation of entropy, S bears the following consequences:
Firstly, $s \propto a^{-3}$, since $V \propto a^3$. this implies, the number of particles in a comoving volume N_i is proportional to the number density n_i divided by the entropy density,

$$N_i \propto \frac{n_i}{s}$$

If the particles are neither created nor destroyed, then $n_i \propto a^{-3}$ and N_i is constant. This is precisely what happened for the total baryon number after baryogenesis,

$$n_B/s = \frac{(n_b - n_{\bar{b}})}{s}$$

Secondly, with the aid of eqn.(24), we obtain

$$g_{*S}(T)T^3a^3 = constant$$

implying

$$T \propto g_{*S}^{-1/3}(T)a^{-1}$$

Away from particle mass thresholds g_{*S} is approximately constant and $T \propto a^{-1}$, as expected. The factor of $g_{*S}^{-1/3}$ accounts for the fact that whenever a particle species becomes non-relativistic and disappears, its entropy is transferred to the other relativistic species still present in the thermal plasma, causing T to decrease slightly less slowly than a^{-1}.

3 Decoupling of the Neutrinos

Neutrinos are coupled to the thermal bath via weak interaction processes,

$$\nu_e + \bar{\nu}_e \leftrightarrow e^- + e^+,$$

$$e^- + \nu \leftrightarrow e^- + \nu$$

The cross section for these interactions as already encountered, $\sigma \sim G_F^2 T^2$ and hence the transition rate, $\Gamma \sim G_F^2 T^5$. As the temperature decreases, the interaction rate drops much more rapidly than the Hubble rate, $H \sim T^2/M_{Pl}$, and we notice,

$$\frac{\Gamma}{H} \sim \left(\frac{T}{1Mev}\right)^3$$

.

We conclude that neutrinos decouple around 1 MeV. (A more accurate computation presented in the annexe gives $T \sim 3.54MeV$).

After decoupling, the neutrinos move freely along geodesics and preserve to an excellent approximate the *relativistic* Fermi-Dirac distribution (even after they become non-relativistic at later times).

With the aid of standard results of *General Theory of Relativity* and dotted with the *Friedmann-Robertson-Walker* metric, we notice that assuming the neutrinos massless, the magnitude of $3 - momentum$, $p \propto a^{-1}$. For convenience, we define a time-independent parameter, $q \equiv ap$ and so, the neutrino number density scales as,

$$n_\nu \propto a^{-3} \int d^3q \frac{1}{e^{q/aT_\nu} + 1}$$

After decoupling, particle number conservation requires, $n_\nu \propto a^{-3}$. The neutrinos following relativistic Fermi-Dirac distribution implies, $T_\nu \propto a^{-1}$. As long as the photon temperature T_γ scales in the same way, we still have $T_\gamma = T_\nu$. However, particle annihilations (specifically, e^-e^+) causes a deviation from $T_\gamma \propto a^{-1}$ in the photon temperature.

3.1 Consequences of Electron-Positron Annihilation

Shortly after the neutrinos decouple, the temperature drops below the electron mass and electron-positron annihilation occurs,

$$e^- + e^+ \leftrightarrow \gamma + \gamma$$

The energy density and entropy of the electrons and positrons are transferred to the photons, but not to the decoupled neutrinos. The photons are thus "heated" (the photon temperature does not decrease as much) relative to the neutrinos. To quantify this effect, we consider the change in the effective number of degrees of freedom in entropy. If we neglect the neutrinos and other decoupled species, we are left with

$$g_{*S}^{ther} = 2 + \frac{7}{8}.4 = \frac{11}{2} \qquad T \gtrsim m_e \tag{25}$$

$$= 2 \qquad T < m_e \tag{26}$$

Since, in equilibrium $g_{*S}^{ther}(aT^3)$ remains constant, we find that aT_γ increases after electron-positron annihilation, $T < m_e$, by a factor $(11/4)^{\frac{1}{3}}$, while aT_ν remains the same. That is, the temperature of neutrinos is slightly lower than the photon temperature after e^+e^- annihilation,

$$T_\nu = \left(\frac{4}{11}\right)^{\frac{1}{3}} T_\gamma \tag{27}$$

For $T << m_e$, the effective number of degrees of freedom of the relativistic species (in energy density and entropy) therefore is,

$$g_* = 2 + \frac{7}{8}.2N_{\mathit{eff}}\left(\frac{4}{11}\right)^{\frac{4}{3}} = 3.36 \tag{28}$$

$$g_{**S} = 2 + \frac{7}{8}.2N_{\mathit{eff}}\left(\frac{4}{11}\right) = 3.94 \tag{29}$$

where we have introduced the parameter N_{eff} as the *effective* number of neutrino species in the universe. If neutrino decoupling was instantaneous then we would have, $N_{\mathit{eff}} = 3$. However, neutrino decoupling was not quite complete when e^+e^- annihilation began, so some of the energy and entropy did leak to the neutrinos. To get the precise value of N_{eff} one also has to consider the fact that the neutrino spectrum after decoupling deviates slightly from the Fermi-Dirac distribution. This spectral distortion arises because the energy dependence of the weak interaction causes neutrinos in the high-energy tail to interact more strongly. Taking this into account, raises the effective number of neutrinos to

$$N_{\mathit{eff}} = 3.046$$

.

3.2 A brief qualitative analysis of Recombination and its consequence

At temperatures above about $1\,eV$, the universe still consisted of a plasma of free electrons and nuclei. Photons were tightly coupled to the electrons via

Compton scattering,

$$e^- + \gamma \leftrightarrow e^- + \gamma$$

which in turn strongly interacted with protons via Coulomb scattering,

$$e^- + p^+ \leftrightarrow H + \gamma$$

. There was very little neutral hydrogen. When the temperature became low enough, the electrons and nuclei combined to form neutral atoms (termed as, *recombination*), and the density of free electrons fell sharply. The photon mean free path grew rapidly and became longer than the horizon distance. The photons decoupled from the matter and the universe became transparent. Today, these photons are the reason for the cosmic microwave background.
With the aid of Boltzmann Equation (in non-equilibrium thermodynamics) and Saha equation therefrom, we obtain the recombination temperature, the temperature at which 90 percent of the electrons have combined with protons to form hydrogen, given by,

$$T_{rec} \approx 0.3eV \simeq 3600K$$

The reason that $T_{rec} << B_H$, the binding energy of hydrogen (which is $13.6\,eV$) is that there are very many photons for each hydrogen atom ($\eta \sim 10^{-9} << 1$), even when $T \lesssim B_H$, the high-energy tail of the photon distribution contains photons with energy, $E > B_H$ so that they can ionize a hydrogen atom.
Using the relation, $T_{rec} = T_0(1 + z_{rec})$, and $T_0 = 2.73K$, we obtain the redshift of the recombination to be

$$z_{rec} \approx 1320$$

.

We define here, the redshift parameter as the fractional shift in wavelength of a photon emitted by a distant galaxy at time t_1 and observed on Earth today $t = t_0$,

$$z \equiv \frac{\lambda_0 - \lambda_1}{\lambda_1}$$

From Quantum Mechanics, we obtain, $\lambda = \frac{h}{p}$. With $p \propto a^{-1}$, we have, for a light emitted at time t_1 with wavelength λ_1 will be observed at t_0 with wavelength

$$\lambda_0 = \frac{a_(t_0)}{a_(t_1)}\lambda_1$$

. And hence, we notice

$$1 + z = \frac{a(t_0)}{a(t_1)}$$

As matter-radiation equality happened at $z \approx 3500$, we conclude that recombination occurred in the matter-dominated era. Using $a_t = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}$ (upon assuming, $a(t_0) = 1$) we obtain an estimate for the time of recombination,

$$t_{rec} \approx \frac{t_0}{(1 + z_{rec})^{3/2}} \sim 290\ 000\,years.$$

Besides, photons and electrons decouple roughly when the interaction rate becomes smaller than the expansion rate,

$$\Gamma_\gamma(t_{dec}) \sim H(T_{dec})$$

Calculation shows that,

$$T_{dec} \sim 0.27\ eV$$

$$z_{dec} \sim 1100$$

$$t_{dec} \sim 380\,000\ years$$

3.3 Cosmic Neutrino Background Radiation and the constraints over Neutrino mass

The relation between T_γ and T_ν holds for the present universe as well. The cosmic neutrino background radiation has therefore a temperature, $T_{\nu,0} = 1.95K = 0.17\ meV$ slightly lower than the cosmic microwave background radiation, $T_{\gamma,0} = 2.73K = 0.24\ meV$. Using, eqns.(14) and (15), we obtain

$$n_\nu = \frac{3}{4} N_{eff} \frac{4}{11} n_\gamma$$

With the magnitude of n_γ at our disposal, we obtain,

$$n_\nu \approx 336\ cm^{-3}$$

The present energy density of neutrinos depends on whether the neutrinos are relativistic or non-relativistic today. Assuming the neutrinos massless, and thereby using eqns.(16) and (17), we obtain,

$$\rho_\nu = \frac{7}{8} N_{eff} \left(\frac{4}{11}\right)^{4/3} \rho_\gamma$$

With the values of ρ_γ and $\rho_{crit,0}$ known, we find out,

$$\Omega_\nu h^2 \approx 1.7\ 10^{-5}$$

However, neutrino oscillation experiments have shown that neutrinos do have mass. The minimum sum of the neutrino masses is $\sum_i m_{\nu,i} > 60\ meV$.

Massive neutrinos behave as radiation-like particles in the early universe, and as matter-like particles in the late universe. Energy density of massive neutrinos, $\rho_\nu = \sum_i m_{\nu i} \rho_{\nu i}$ corresponds to $\Omega_\nu h^2 \approx \frac{\sum_i m_{\nu,i}}{94\ eV}$.

Measurements of tritium $\beta - decay$ find that $\sum_i m_{\nu i} < 6\ eV$. Besides, observations of the cosmic microwave background, galaxy clustering and type Ia supernovae together put an even stronger bound, $\sum_i m_{\nu i} < 1\ eV$. This signifies that although neutrinos contribute at least 25 times the energy density of photons, they are still a sub-dominant component overall, $0.001 < \Omega_\nu < 0.02$.

3.4 Proposition of a fourth generation neutrino: Motivation and Consequences

In this section, we implicitly assume that the neutrinos are relativistic at present and perform the relativistic treatments of different physical observables.
As already viewed, the energy density of the Universe is given by,

$$\rho = \frac{\pi^2}{30}(g_\gamma T_\gamma^4 + \frac{7}{8} g_\nu T_\nu^4)$$

which gives,

$$\rho = \frac{\pi^2}{30} g_\gamma T_\gamma^4 \left[1 + \frac{7}{8}\frac{g_\nu}{g_\gamma}\left(\frac{T_\nu}{T_\gamma}\right)^4\right]$$

The ratio, $\frac{g_\nu}{g_\gamma}$, as predicted by the Standard Model should be 3. However, observational evidences ensure the value to be 3.2.
There is no ambiguity on the internal degrees of freedom of photons and the Standard Model neutrinos. This opens therefore an opportunity to postulate a neutrino of the fourth generation to account for the observed mismatch which decoupled at the same temperature as the neutrinos. On the other hand, the Standard Model being complete (it has exactly three generation of fermions, confirmed by the standard deviation of measured masses of vector bosons) the fourth generation neutrino must be accommodated in a Beyond Standard Model scenario.
Accepting its existence, let us characterize this new neutrino, which we call ν'. We propose that, this particle interacts only weakly just like the Standard Model neutrinos, this implies that both ν and ν' decoupled from the plasma at the same instant.
From the very reason for its postulation, it follows that ν' is still relativistic at present and so we could have an upper bound for its mass.
Since both ν and ν' decouple at same instant of time and streams through ever since being relativistic entities, we must have

$$T_\nu = T_{\nu'}$$

We notice in parallel with these traits that, since during recombination, the internal degrees of freedom of neither ν nor ν' were involved in the evaluation of the scaling of T_γ,

$$T_{\nu,0} = T_{\nu',0} = \left(\frac{4}{11}\right)^{\frac{1}{3}} T_{\gamma,0}$$

However, if we accept the fractional value of $g_{\nu'}$ $(= 0.2)$, with the same approach, as we have followed to determine n_ν above, we obtain

$$n_{\nu'} \approx 22\ cm^{-3}$$

4 Annexe

4.1 Evaluation of the precise temperature of neutrino decoupling

Neutrinos decouple chemically in the radiation dominated epoch of the Universe. The decoupling temperature is determined by the equation, $H(T_{dec}) \sim \Gamma(T_{dec})$. Now, $\Gamma_\nu = n_\nu < \sigma v >$, and $n_\nu = \frac{3}{4}\frac{\zeta(3)}{\pi^2} g_\nu T^3$. In order to compute $< \sigma v >$ we use few notions of Quantum Field Theory.
We consider the reactions,

$$\nu + \bar{\nu} \leftrightarrow e^- + e^+$$

$$\nu + e^- \leftrightarrow \nu + e^-$$

in the s-channel under low energy approximation.
The Lagrangian of the first of the above processes is given by,

$$L = \frac{G_F}{\sqrt{2}} \bar{\nu}\nu\bar{e}e$$

where ν and e are Dirac spinors while $\bar{\nu}$ and $\bar{e}$ are adjoint Dirac spinors.
Neglecting the neutrino mass, $m_\nu \approx 0$, we obtain,

$$< \sigma v > \simeq \frac{1}{symmetry\ factor} \frac{1}{2s_1 + 1} \frac{1}{2s_2 + 1} \int \frac{|M|^2}{16\pi s} d\cos\theta$$

where s_1 and s_2 are the spins of ν and $\bar{\nu}$ respectively. The symmetry factor for this process is 1.
The *matrix element* M of the reaction is given by,

$$M = \int d^4x < \bar{e}e|iL|\bar{\nu}\nu >$$

Expressing the Dirac and the adjoint Dirac spinors as the Fourier Transforms of the creation and annihilation operators acting on momentum space, we obtain,

$$M = i\frac{G_F}{\sqrt{2}} \bar{v}_\nu(p_2)u_\nu(p_1)\bar{u}_e(p_3)v_e(p_4)$$

where u and v are Dirac spinors in momentum space associated with annihilation and creation respectively and we implicitly sum over all the possible spin states.
p_1, p_2, p_3 and p_4 are the $4 - momenta$ of ν, $\bar{\nu}$, e^- and e^+ respectively.
Considering the operation, element par element of the spinors, we get

$$M = i\frac{G_F}{\sqrt{2}} \bar{v}_{\nu_a}(p_2)u_{\nu_a}(p_1)\bar{u}_{e_b}(p_3)v_{e_b}(p_4)$$

where the summation is done over the dummy indices. Similarly,

$$M^* = -i\frac{G_F}{\sqrt{2}} \bar{v}_{e_c}(p_4)u_{e_c}(p_3)\bar{u}_{\nu_d}(p_1)v_{\nu_d}(p_2)$$

This gives, on rearranging

$$|M|^2 = \frac{G_F^2}{2} \; (\bar{u}_{\nu_d}(p_1)u_{\nu_a}(p_1)) \; (\bar{v}_{\nu_a}(p_2)v_{\nu_d}(p_2)) \; (\bar{u}_{e_b}(p_3)u_{e_c}(p_3)) \; (\bar{v}_{e_c}(p_4)v_{e_b}(p_4))$$

Summing over all possible spin states, using the properties of γ matrices, we finally obtain,

$$|M|^2 = \frac{G_F^2}{2} 2s(2s - 8m_e^2)$$

Now, using the standard result for such a reaction,

$$\frac{d\sigma}{d\Omega} = \frac{1}{Symmetry\ factor} \frac{1}{2s_1+1} \frac{1}{2s_2+1} \frac{|M|^2}{64\pi^2 s} \frac{\sqrt{s - 2m_{e^-}^2 - 2m_{e^+}^2 + \left(\frac{m_{e^-}^2 - m_{e^+}^2}{s}\right)^2}}{\sqrt{s - 4m_\nu^2}}$$

With $v_\nu \approx c = 1$, we finally get,

$$< \sigma v >= \frac{G_F^2}{32\pi} \frac{(s - 4m_e^2)^(3/2)}{\sqrt{s}}$$

With $s = 4E_\nu^2$, we have

$$< \sigma v >= \frac{G_F^2}{8\pi} \frac{(E_\nu^2 - m_e^2)^{(3/2)}}{E_\nu}$$

where E_ν is the energy of ν in the centre of mass frame of the interaction.
At relativistic limit, the mean energy per fermionic particle is given by,

$$E_f = \frac{7\pi^4}{180\zeta(3)} T \approx 3.151T$$

This gives,

$$< \sigma v >= \frac{G_F^2}{8\pi} \left(\frac{7\pi^4}{180\zeta(3)}\right)^2 T^2 \simeq \frac{9.93G_F^2}{8\pi} T^2$$

Therefore we finally obtain,

$$\Gamma_{\nu\bar{\nu}\leftrightarrow e^- e^+} = n_\nu \frac{9.93G_F^2}{8\pi} T^2$$

In the radiation dominated Universe, $H = \frac{1.66 g_*^{1/2} T^2}{M_{Pl}}$
The constraint, $H(T_{dec}) \sim \Gamma(T_{dec})$ yields, with $g_* = 10.75$ and $g_\nu = 2$ since we consider a particular flavour of neutrinos for the instant

$$T_{dec} \sim 3.54\ MeV$$

4.2 Modification of the decoupling temperature of neutrinos for the presence of ν'

With the existence of the 4^{th} generation neutrinos, $g_*(T_{dec})$ would increase to 11.1, this causes an increase in the magnitude of $T_{\nu,dec}$ (by a factor of $\left(\frac{11.10}{10.75}\right)^{1/6} \sim$ 1.004). But ν' bearing a fractional number of internal degrees of freedom, $T_{\nu',dec} > T_{\nu,dec}$, ($T_{\nu',dec} = 5^{1/3}\left(\frac{11.10}{10.75}\right)^{1/6} T_{\nu,dec} \sim 1.200$ times the modified $T_{\nu,dec}$). However, the relation $T_{\nu,0} = T_{\nu',0}$ still holds true, because although ν' decouples at a temperature higher than that of ν, ν' and the thermal plasma still shares identical temperature until the electrons and the positrons decouple kinetically. Thus, it turns out that, it is coherent to propose the existence of another generation of neutrinos accounting for the mismatch of the ratio g_ν/g_γ, which have decoupling temperature different from that of the Standard Model neutrinos, although their present temperature being identical.

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